

3 Elementary bifurcations

3.1 Examples

We already saw in the previous lecture that mathematical models in the form of ODE often depend on parameters. Moreover, when the parameter changes, the behavior of the solutions to ODE sometimes suddenly changes as well; it is said that a bifurcation occurs. In this lecture I will consider simplest possible bifurcations in the first order autonomous ODE

$$\dot{x} = f(x, \alpha), \quad x \in U \subseteq \mathbf{R}, \alpha \in \mathbf{R}, \quad (1)$$

where the explicit dependence on the parameter α is shown to emphasize that we study the behavior of solutions when the parameter values vary.

Example 1 (*Fold or saddle-node bifurcation*). Consider the following ODE:

$$\dot{x} = \alpha + x^2 =: f(x, \alpha). \quad (2)$$

The equilibria of this equation are determined by the equation

$$\alpha = -x^2,$$

which has two solutions

$$\hat{x}_{1,2} = \pm\sqrt{-\alpha},$$

if $\alpha < 0$, has one solution

$$\hat{x} = 0,$$

if $\alpha = 0$, and no solutions if $\alpha > 0$. The stability of these solutions is also easy to determine: \hat{x}_1 (the one with the positive sign) is unstable and \hat{x}_2 (the one with the negative sign) is asymptotically stable. When $\alpha = 0$ the equilibrium $\hat{x} = 0$ becomes non-hyperbolic, since in this case $f'_x(0, 0) = 0$ (in case when a function depends on more than one variable and I use prime to denote differentiation, the subscript indicates the variable with respect to which the derivative is taken). The whole picture can be described as “as parameter α increases two equilibria approach each other, collide, and disappear.” It is convenient to summarize the analysis in the *bifurcation diagram*, which is shown in the figure. In this case, when we deal with a scalar equation and the parameter is one dimensional, the bifurcation diagram can be presented as a direct product of the parameter space and the phase space, $\mathbf{R} \times \mathbf{R}$. The equation $f(x, \alpha) = 0$ determines the set of equilibria (in the considered case — parabola $\alpha = -x^2$). Projection of this set on the α -axis has a singularity at the point $(0, 0)$ at which the bifurcation occurs: two equilibria turn into one and disappear. This bifurcation is called in the literature *fold*, or *tangent*, or *saddle-node* bifurcation. I repeat that this bifurcation happens when $\lambda = f'_x(0, 0) = 0$, i.e., when our equilibrium in the system is non-hyperbolic.

Here is another example.

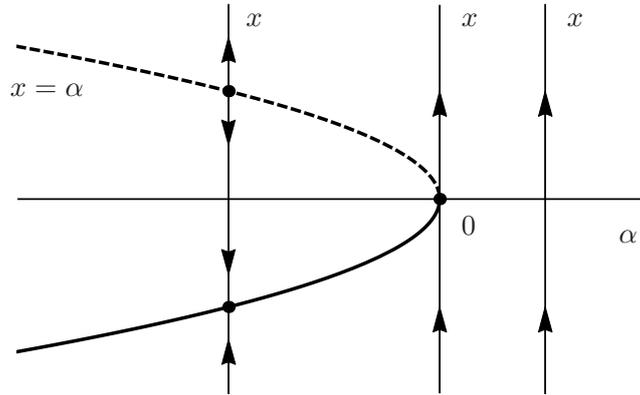


Figure 1: The bifurcation diagram of $\dot{x} = \alpha + x^2$ (fold bifurcation). The bold curve corresponds to the stable equilibrium, the dashed curve corresponds to the unstable equilibrium. Three phase portraits are shown: one with $\alpha < 0$, one with $\alpha = 0$, and one with $\alpha > 0$. The shaded circles show the equilibria of the system, and the arrows indicate the direction of the phase flow

Example 2 (*Pitchfork* bifurcation). Consider

$$\dot{x} = \alpha x - x^3 =: f(x, \alpha), \quad x \in U \subseteq \mathbf{R}, \alpha \in \mathbf{R}. \quad (3)$$

If $\alpha < 0$ then there is unique asymptotically stable equilibrium $\hat{x} = 0$, if $\alpha > 0$ then additionally we have $\hat{x}_{1,2} = \pm\sqrt{\alpha}$, both of which are asymptotically stable, whereas $\hat{x} = 0$ becomes unstable. When $\alpha = 0$ there is only asymptotically stable $\hat{x} = 0$, which is, however, non-hyperbolic: $f'_x(0, 0) = 0$. The bifurcation diagram is presented in the figure. This bifurcation has the name *pitchfork* bifurcation.

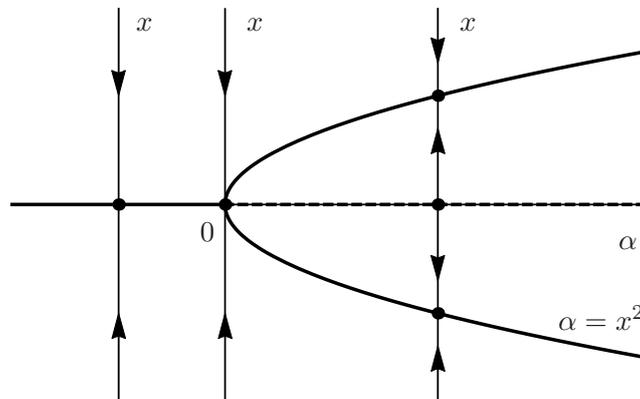


Figure 2: The bifurcation diagram of $\dot{x} = \alpha x - x^3$ (supercritical pitchfork bifurcation). The bold curves correspond to the stable equilibria, the dashed curve corresponds to the unstable equilibrium. Three phase portraits are shown: one with $\alpha < 0$, one with $\alpha = 0$, and one with $\alpha > 0$. The shaded circles show the equilibria of the system, and the arrows indicate the direction of the phase flow

The considered examples are very illuminating. However, note that I did not actually define what bifurcation is, and what are the conditions that determine one or another bifurcation. The general

definitions are not actually needed at this point, therefore I will discuss the definition of a bifurcation specific to the first order ODE (1).

First, I need to understand which ODE are considered to be different, and what I mean when I say that “the behavior of solutions changes.” That is, I need to obtain the means to compare two different ODE of the form (1). Comparison of any mathematical objects is based on an equivalence relation, which allows to identify classes of equivalent objects and study the relationships between these classes.

Definition 3. *Consider two ODE*

$$\begin{aligned}\dot{x} &= f(x), & x \in U \subseteq \mathbf{R}, \\ \dot{x} &= g(x), & x \in V \subseteq \mathbf{R}.\end{aligned}$$

They are called topologically equivalent, if they have equal number of equilibria of the same stability, located in the same order on the phase line.

In short, we call two ODE of the form (1) with fixed parameter values *topologically equivalent*, if they have the same structure of the phase space. The phase portraits of topologically equivalent ODE are also called topologically equivalent.

Definition 4. *Appearance of topologically non-equivalent phase portraits under variation of the parameters is called bifurcation.*

Note that in my case of scalar ODE, any bifurcation is associated with the change of the number or stability of the equilibria of the system. Therefore, we consider so far only *bifurcations of equilibria*. The exact value of the parameter at which a bifurcation occurs is called the *bifurcation value* or *bifurcation point*.

How to actually determine the bifurcation values, if any? If you look again at the examples, you will find that the bifurcations occurred in these systems exactly when one or another equilibrium became non-hyperbolic. It turns out that this is indeed the defining condition for a bifurcation of the equilibrium \hat{x} to happen:

$$f'_x(\hat{x}, \alpha) = 0 \implies$$

α is a bifurcation value. Here is one more example to practice sketching the bifurcation diagram.

Example 5 (*Transcritical bifurcation*). Consider

$$\dot{x} = \alpha x - x^2, \quad x \in U \subseteq \mathbf{R}, \alpha \in \mathbf{R}.$$

Here we always have two equilibria $\hat{x} = 0$ and $\hat{x}_1 = \alpha$, which “collide” at $\alpha = 0$ into one non-hyperbolic equilibrium; at the point $\alpha = 0$ these equilibria exchange stability. The bifurcation diagram is given in the figure below. The bifurcation value is again zero: $\alpha = 0$.

3.2 General discussion

3.2.1 Hyperbolic equilibria are insensitive to small perturbations

First, let me explain why exactly the necessary condition for an equilibrium bifurcation in (1) is non-hyperbolicity of this point. As a warm up I consider the equation

$$\dot{x} = -x,$$

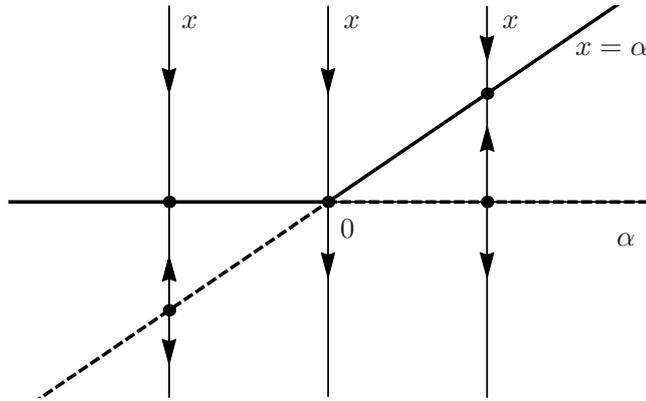


Figure 3: The bifurcation diagram of $\dot{x} = \alpha x - x^2$ (Transcritical bifurcation). The bold curves correspond to the stable equilibria, the dashed curves correspond to the unstable equilibria. Three phase portraits are shown: one with $\alpha < 0$, one with $\alpha = 0$, and one with $\alpha > 0$. The shaded circles show the equilibria of the system, and the arrows indicate the direction of the phase flow

which has a hyperbolic asymptotically stable equilibrium $\hat{x} = 0$ at the origin, and *its parametric perturbation* in the form

$$\dot{x} = \alpha - x,$$

which coincides with the original equation if $\alpha = 0$. It is straightforward to see that the perturbed equation has exactly the same equilibrium with the same stability properties, see the figure.

Now I consider a general ODE of the form

$$\dot{x} = f(x),$$

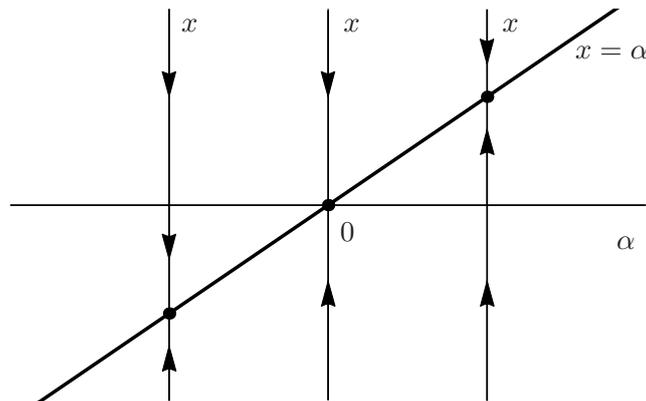


Figure 4: The bifurcation diagram of $\dot{x} = \alpha - x$ (hyperbolic equilibrium is stable with respect to perturbations). The bold curves correspond to the stable equilibria, the dashed curves correspond to the unstable equilibria. Three phase portraits are shown: one with $\alpha < 0$, one with $\alpha = 0$, and one with $\alpha > 0$. The shaded circles show the equilibria of the system, and the arrows indicate the direction of the phase flow

and assume that $f \in \mathcal{C}^1$, $f(0) = 0$, and $f'(0) \neq 0$, i.e., that I have an equilibrium at the origin and that this equilibrium is hyperbolic (note that the requirement to have the equilibrium at the origin is not restrictive because if one considers equilibrium $\hat{x} \neq 0$, it is always possible to make the change of the variables, say $y = x - \hat{x}$, and for y the equilibrium $\hat{y} = 0$ appears). Together with the ODE, I consider its perturbation

$$\dot{x} = F(x, \alpha),$$

where $F: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, and $F \in \mathcal{C}^1$ function satisfying

$$F(x, 0) = f(x), \quad \frac{\partial F}{\partial x}(0, 0) = f'(0) \neq 0. \quad (4)$$

Let me investigate the equilibria of the perturbed equation if $\alpha \neq 0$. These new equilibria probably will be different from zero because generically I will have $F(0, \alpha) \neq 0$.

Conditions (4) together with the *implicit function theorem* (the exact statement is given at the end of the lecture) imply that for small enough α (technically, for $|\alpha| < \delta$) there exist a *unique* \mathcal{C}^1 function $\psi(\alpha)$ with $\psi(0) = 0$ and

$$F(\psi(\alpha), \alpha) = 0.$$

This means that under the perturbation only one equilibrium is possible, and this equilibrium is given by $\hat{x} = \psi(\alpha)$. The stability of this equilibrium is determined by (the sign of)

$$F'_x(\psi(\alpha), \alpha).$$

We know that $F'_x(0, 0) = f'(0) \neq 0$, and since F'_x and $\psi(\alpha)$ are continuous, then the sign of $F'_x(\psi(\alpha), \alpha)$ coincides with the sign of $f'(0)$ for small enough α . Therefore, the stability type of the perturbed equilibrium coincides with that of $\dot{x} = f(x)$. Or, in words, the flow near a hyperbolic curve is insensitive to small perturbations of the equation, no bifurcation is possible. Hence to have a bifurcation, my equilibrium must be non-hyperbolic.

3.2.2 Universality of the typical bifurcations

In the first section of this lecture some examples were presented for the first order parameter dependent ODE. The natural questions are actually how long is the possible list of bifurcations, and what kind of generalizations can be made out of these examples.

It turns out that actually in a *generic* case the only bifurcation which we can see in a system with one parameter is the fold bifurcation. At this point I would not want to go into the precise mathematical details of this statement, and confine to general words. In short, the major tool of analysis of autonomous ODE is the coordinate and parameter changes that put system in the *simplest form* (we need to define of course what the simplest form means in general, for now this can be a polynomial form of the lowest possible degree). This simplest form is called the *normal form*. It turns out that for a general function f in (1) its normal form coincides with the equation from the first example in this lecture. Here is the exact statement of the theorem, whose proof is actually relies heavily on the implicit function theorem and the inverse function theorem.

Theorem 6. *Any differential equation*

$$\dot{x} = f(x, \alpha), \quad \alpha \in \mathbf{R}, \quad (1)$$

that for $\alpha = 0$ has an equilibrium $\hat{x} = 0$ satisfying

$$\begin{aligned}\lambda &= \frac{\partial f}{\partial x}(0, 0) = 0, \\ f''_{xx}(0, 0) &= \frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0, \\ f'_\alpha &= \frac{\partial f}{\partial \alpha}(0, 0) \neq 0,\end{aligned}\tag{5}$$

by variable and parameter changes can be put in the form

$$\dot{y} = \beta + sy^2 + o(y^2), \quad s = \text{sgn}(f''_{xx}(0, 0)),\tag{6}$$

where β is a new parameter.

Moreover, equation (6) is topologically equivalent in a small enough neighborhood of the point $(\beta, y) = (0, 0)$ to one of the following normal forms

$$\dot{y} = \beta \pm y^2.\tag{7}$$

Remark 7.

- The requirements that the equilibrium is at the origin and the bifurcation value of the parameter is zero are not essential because I can always find coordinate and parameter changes to make sure they hold.
- The bifurcation diagram with the sign “+” is given in the first figure in this lecture. The second one is analogous (see your homework problems).
- If the form of f is such that for any α $f(0, \alpha) = 0$ then the second condition in (5) does not hold. If I exchange it for the condition $f''_{x\alpha}(0, 0) \neq 0$, then one obtains the normal form of the transcritical bifurcation

$$\dot{y} = \beta y + sy^3, \quad s = \text{sgn}(f''_{xx}(0, 0)).$$

- If the differential equation is symmetric with respect to the variable x : $-f(-x, \alpha) = f(x, \alpha)$, i.e., if $f(x, \alpha)$ is odd with respect to x , then the second condition in (5) also cannot hold. If one requires that $f'''_{xxx}(0, 0) \neq 0$ then the normal form of the pitchfork bifurcation appears:

$$\dot{y} = \beta y + sy^3, \quad s = \text{sgn}(f_{xxx}(0, 0)).$$

The normal form with the “-” sign was considered in one of the examples. This bifurcation is called *supercritical* (I have two stable equilibria while the origin is unstable). In case of “+”, the bifurcation is *subcritical*: one observes two unstable equilibria while the origin is stable.

3.2.3 Implicit function theorem

Probably the main tool in the bifurcation theory is the implicit function theorem, whose statement in the simplest case when the coordinate is $x \in \mathbf{R}$ and the parameter is $\alpha \in \mathbf{R}$ is given here.

Theorem 8. *Suppose that $F(x, \alpha)$ is a $\mathcal{C}^{(1)}$ function, $F: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, satisfying*

$$F(0, 0) = 0, \quad \frac{\partial F}{\partial x}(0, 0) \neq 0.$$

Then there exists a unique locally defined $\mathcal{C}^{(1)}$ function

$$x = \psi(\alpha),$$

such that

$$\psi(0) = 0, \quad F(\psi(\alpha), \alpha) = 0$$

for all α in a neighborhood of the origin.

Moreover,

$$\psi'_\alpha(0) = -\frac{F'_\alpha(0, 0)}{F'_x(0, 0)}.$$